Breakup of two-dimensional into three-dimensional Kadomtsev-Petviashvili solitons

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This paper reports on three-dimensional simulations that follow exact, z-symmetric soliton solutions to an important model equation of plasma physics and superfluid helium (Bose condensate). This is the Kadomtsev-Petviashvili equation. Solitons are seen to break up when perturbed along z. Dependence of growth on the wave number of the perpendicular perturbation is found numerically. This leads to a wave number producing the maximum rate of breakup. Due to numerical instabilities, a somewhat smaller wave number must be used. Fully three-dimensional entities are produced. After a while they become virtually identical to known, azimuthally symmetric solutions. Based on this, implications for the reconnection hypothesis formulated by Feynman, used in superfluid helium II theory, are indicated. [S1063-651X(98)15905-6]

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Solitons are sometimes described as the classical counterparts of elementary particles. They are well researched compact entities that appear both in nature and in present day mathematical considerations of nonlinear, partial differential equations.

The three-dimensional (3D) Kadomtsev-Petviashvili equation considered here [1] is

$$(n_t+6nn_x+n_{xxx})_x-3(n_{yy}+n_{zz})=0. \qquad (1)$$

This equation is known as KPI in three dimensions. It describes the dynamics of solitons and nonlinear waves in at least two media (plasmas and superfluids) [2–4] about which more follows. Equation (1) is usually much simpler than the full set of equations it models. In deriving it for propagating phenomena, one assumes weak dispersion and that the soliton or nonlinear wave in question propagates along the x axis. Changes in y and z are slower than in the direction of motion. Equation (1) is integrable by inverse scattering when $\partial_z=0$. For an extensive discussion of the derivation of (1) and some solutions, see Ref. [5], Chaps. 5 and 8.

The KPI equation appears in plasma physics when describing small amplitude, fast magnetosonic (FMS) waves propagating in a low-β $(=8\pi p/B^2)$ magnetized plasma. If these waves propagate at an angle with respect to the magnetic field, collapse may occur. Collapse mechanisms can, in turn, lead to transfer of energy to the plasma ions. A better understanding of these mechanisms is lacking. Certain restrictions, such as those of small amplitude, long wavelength, propagation velocity within a small interval around that for linear FMS waves, wave frequency small as compared with the ion cyclotron frequency, etc., are assumed when deriving KPI in this context.

A second area where three-dimensional KPI appears is the condensate model of superfluid helium. The two- and three-dimensional solitons of KPI then model degenerate limits of the vortex lines and rings, respectively, that have been observed there. In this context, KPI is derived as a limit of the Nonlinear Schrödinger equation with cubic nonlinearity, which itself is a somewhat controversial though popular model for superfluid helium II. (The KPI limit is obtained for entities moving with velocities slightly below that of infinitesimal sound waves (called phonons.) At present, it would seem that KPI is more firmly grounded in physical reality in the case of FMS waves as compared to the Bose condensate context. Nevertheless, we will come back to this latter, fascinating context.

It is of interest when a line soliton distorts in three dimensions and produces a known nonlinear structure. In the superfluid-helium context, this would correspond, if Eq. (1) is to be accepted as a model, to a step toward answering the question of whether a line vortex pair of opposite polarity can break up and recombine into an array of vortex rings (or else, say, one vortex ring and two U-shaped vortices). This is a crucial question in He II theory.

Exact N-soliton solutions to both one- and two-space dimensional versions of Eq. (1) are well known. The one-dimensional soliton, $n(x-vt)$, is unstable in two dimensions [5] and its breakup into an array of two-dimensional, exact-soliton solutions has been demonstrated both analytically and numerically [6–8]. The two-dimensional soliton, $n(x-vt,y)$, is stable in two dimensions [9].

It is natural to ask the generally more physical question of what happens in three dimensions. As the 1D soliton, having plane symmetry, is known to disintegrate into 2D entities in two dimensions, it should certainly break up somehow in three dimensions. The 2D soliton, $n(x-vt,y)$, is known from theory to be unstable in three dimensions [9] (see the Appendix). However, perturbing it along the z axis in a simulation, actually seeing it break up and following the debris all the way to a possible 3D structure, demands somewhat cumbersome numerics. As far as we know, this has not yet been done. Instability of 3D, azimuthally symmetric solitons *has* been investigated numerically [4,10–12]. In contradistinction to the instability of the 2D soliton, which will be seen to lead to complete destruction and formation of new 3D structures, the instability in question was found to lead to gradual collapse. This involves a steepening and narrowing, but with some structure essentially conserved until a final implosion on the axis occurs. Some theoretical estimates of the general behavior involved can be found in [4] (first reference).

It might seem odd that simulations for distortion of the 3D soliton have been performed, whereas such simulations for the 2D one have not. The explanation is in the symmetry. The 3D soliton behavior was studied in x, ρ $[=(y^2+z^2)^{1/2}]$ space, neglecting θ dependence. On the other hand, disintegration of the 2D soliton in three-dimensional space can only be followed in a fully three-dimensional simulation, which is much more time consuming. This task is the subject of the present work.

Although we know from theory that our 2D soliton will disintegrate, there are at least three further questions to be addressed.

(1) What is the wave number of the perpendicular perturbation leading to the shortest lifetime of the 2D soliton in three dimensions? (2) When this or a similar perturbation is applied, will any products of the inevitable breakup be robust? (3) If the answer to question (2) is affirmative, will any robust fragments evolve to the 3D solutions, known approximately from the theory of [13]?

Shortly, we will answer question (1) with numerical simulations. First, however, a few brief comments on questions (2) and (3) are in order. Recently, a similar 3D simulation was performed for a different model soliton equation [Zakharov-Kuznietsov, which differs from (1) only in the y and z terms [14]]. There, the answers to both questions were affirmative. However, in that case, the 3D solitons were known to be stable, in contradistinction to the present study of Eq. (1). Thus, until we performed the simulation, we had to consider both questions, (2) and (3), as open.

The two-dimensional soliton solution to Eq. (1) [15] is

$$n(x,y,t)=\frac{4\nu[1-\nu(x-3\nu t)^2+\nu^2y^2]}{[1+\nu(x-3\nu t)^2+\nu^2y^2]^2}, \quad \nu>0. \qquad (2)$$

Note that n, which is the *excess* over the mean density in (1), can be negative. Valleys appear around the x axis. As already mentioned, theory tells us that this soliton is unstable in x,y,z,t. In the liquid helium context, Eq. (2) is the KPI limit of an oppositely directed vortex pair solution of the nonlinear Schrödinger equation.

The three-dimensional soliton somewhat resembles Eq. (2) rotated around the x axis. However, known solutions are approximate [there is no exact formula resembling Eq. (2) with $y\rightarrow\rho$] [13]. This is not surprising, in view of the nonintegrability of Eq. (1) in 3D. It is possible, however, to find the form of n in the far field by concentrating on the linear terms in Eq. (1). We find

$$n\rightarrow\text{const}\times\frac{\nu\rho^2-2(x-3\nu t)^2}{[\nu\rho^2+(x-3\nu t)^2]^{5/2}}. \qquad (3)$$

The complete solution was found numerically in [13] as a limit of the nonlinear Schrödinger equation solution (NLS was solved by Chebyshev-Legendre series expansion). The constant density lines in x,ρ space somewhat resemble those drawn from Eq. (2) for x,y [compare Fig. 1(d) of Ref. [13] with the bottom trace of our Fig. 1(a)]. The resemblence is only qualitative. For example, large-denominator behavior from Eq. (2) is

$$n\rightarrow\text{const}\frac{\nu y^2-(x-3\nu t)^2}{[\nu y^2+(x-3\nu t)^2]^2}. \qquad (4)$$

Because of the geometry, a coefficient in the numerator and the power of the denominator are different from those of Eq. (3), but general behavior is similar.

Our calculations were performed on both a HP Apollo 9000 model 720 workstation and on a Cray EL98. The numerical algorithm used for calculating the time evolution was the leapfrog algorithm, applied when Eq. (1) was integrated over x. Periodic boundary conditions were assumed. More about the algorithm and its stability can be found in the second of Ref. [8]. These 3D calculations took about a thousand computer hours.

The soliton (2) was wiggled along z such that, at $t=0$, x was replaced by $x+\delta\cos(k_z z)$. We do not know the value of k_z, corresponding to the maximum growth rate of the instability, from theory. Therefore, this critical ''wavenumber'' was found numerically. The result is shown in Fig. 2 (for convenience, we actually used a smaller value of k_z in our subsequent simulations, to avoid numerical instabilities [8]). The behavior for very small k_z in the figure is indicated by an expansion calculation; see the Appendix. The two methods are complementary and give consistent results.

The Kadomtsev-Petviashvili equation is peculiar in that even the initial conditions must fulfill an infinite set of constraints. The most obvious one follows from an x integration:

$$\nabla_\perp^2\int_{-\infty}^{\infty}n\ dx=0. \qquad (5)$$

Our initial condition satisfies all these constraints. Again, this aspect is discussed extensively in [8] (for 2D; see also [16] for theory). One of the conclusions of Ref. [8] is that in fact not satisfying them had little impact on the result. However, here we do satisfy these constraints. Figure 1 shows three-dimensional visualizations of three constant-n surfaces as the perturbed soliton propagates. Two cross sections augment each frame. The message is that the 2D soliton does break up when perturbed along the third direction, producing structures some of which are seen changing their symmetry from z to θ, though at different rates.

Note how little the x,y plane traces change during the simulation. This was suggested by the similarity between Eqs. (3) and (4). At the same time, the x,z traces are changing rapidly so as to mimic the contemporary ones for x,y. After a while, a 3D soliton results from some of the fragments, rather like a butterfly emerging from a cocoon. This is confirmed by the cross sections. There is a recognizable residue of the 2D soliton trailing behind it. The new 3D soliton moves forward faster than both the parent 2D soliton and its residue.

The two cross sections of the 3D soliton are similar and resemble those found numerically in Ref. [13]. Comparison should be made between their $1-\rho$ for the fourth frame of Fig. 2 of [13] and our n. (Their parameter $U=0.69$, whereas KPI corresponds to small but positive $2^{-1/2}-U$, as indeed it is for this U value, for which it is 0.017.)

It is hard to say how the space array of 3D solitons will depend on the boundary conditions in general. In our calculation, we obtained one emerging 3D soliton per box, thus the space period was just the height of the box. However, we cannot rule out the possibility that, with very different k_z, and hence different height, two or more 3D solitons could be

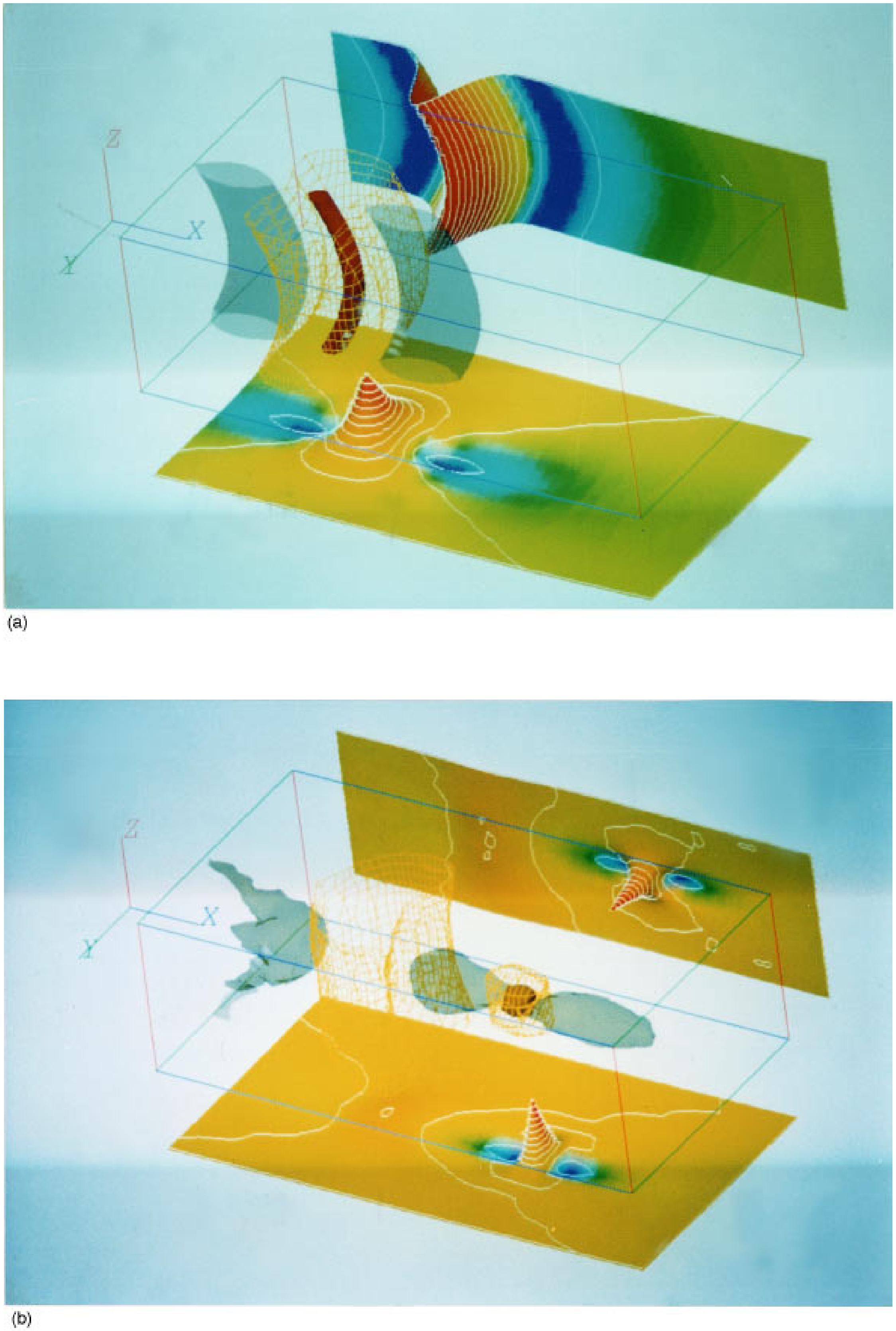
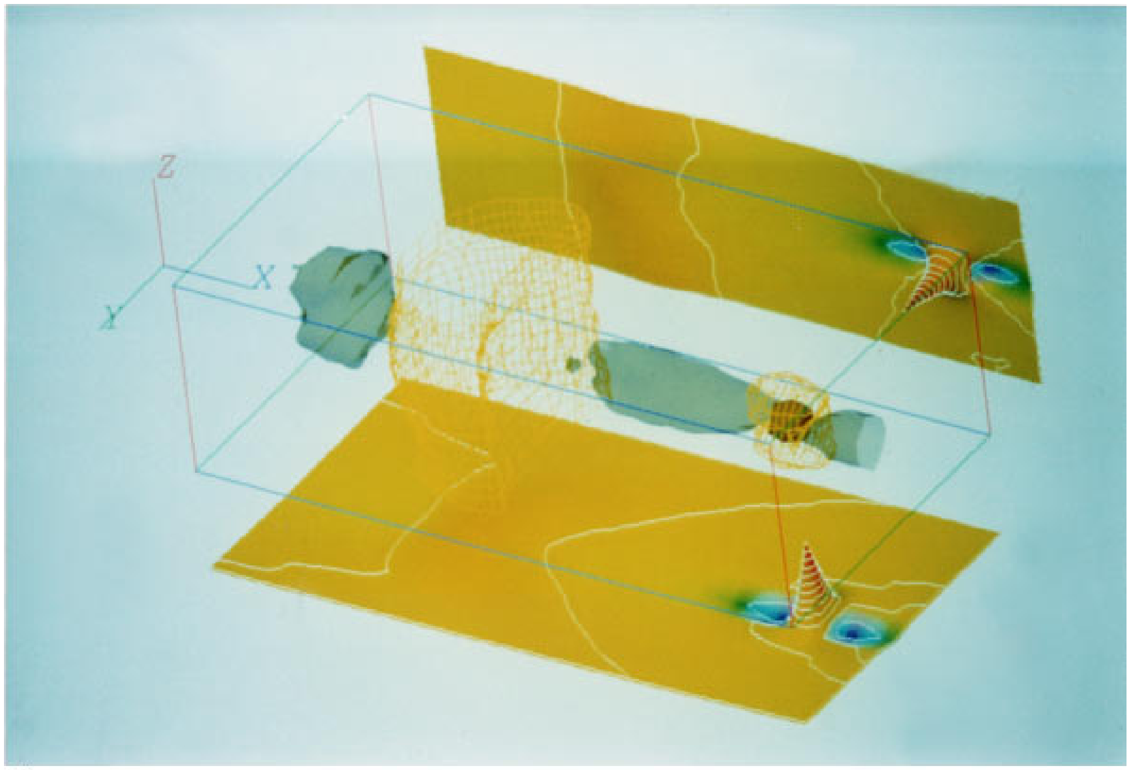


FIG. 1. (Color) Consecutive stages of the evolution of a 2D soliton initially perturbed along the z axis. The equation of the perturbed soliton at $t=0$ is given by Eq. (2) with x replaced by $x+\delta\cos(k_z z)$. Here $\nu=\frac{17}{6}$, $\delta=0.12$, $k_z=1.4$. Three surfaces of constant n $(-0.8, 1, 8.84)$ are seen. They are for $t=0$, $t=0.094$, and $t=0.140$. The 2D soliton breaks up, producing new structures and debris. Some of the robust fragments are changing their symmetry to finally produce a 3D soliton. Cross sections along x,y (bottom) and x,z (top) are shown. Initially the bottom ones correspond to constant-n lines as given by Eq. (2). For halftones of stages intermediate between the first and second frames, see [17].



(c)

FIG. 1 (*Continued*).

generated per box. Thus all we can say is that the z period will in general be just the height of the box divided by n, where n is a natural number. Just how n will depend on k_z is an open question.

Recently the present authors prematurely published results of a simulation that ended before the 3D soliton was formed. We erroneously surmised that this entity would probably not put in an appearance [17]. There is now no doubt that it does (the present simulation ran for twice as long as that of [17]). According to [4,11], it should gradually collapse, steepening and narrowing. Our computational capabilities did not even permit us to observe the onset of this phenomenon (remember, the simulations of [4,13] were θ independent, whereas ours are not). All we can say is that, on the time scale of the dynamics we have observed, collapse must be very gradual.

With all the reservations mentioned above (nonlinear Schrödinger being a controversial model for superfluid helium, KPI being just a limit of NLS), our results nevertheless strongly suggest that an oppositely polarized pair of line vortices in ^{4}He II can coalesce and then reconnect into an array of ring vortexes.

Up to now, reconnection, resulting in the formation of small ring vortices out of a pair of oppositely directed line vortices (or two oppositely directed nearby sections of one large distorted circular vortex) was simply postulated in ^{4}He II theory [18,19]. Now we have a step-by-step indication that this metamorphosis indeed takes place. This is just an indication, as the reconnection was found for the KPI limit, at which NLS vortices degenerate. However, until it proves possible to isolate two opposed line vortices in ^{4}He II and see them reconnect in an experiment, any theoretical strengthening of the reconnection hypothesis should be of some importance. (Such reconnection has been observed in a very different context, the vortex trail of a B-47 aircraft [20]. Theoretical work has been done in regular fluid dynamics [21].)

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APPENDIX

The general nature of the onset of instability can be investigated theoretically for small k_z [5]. We expand in k_z and assume that the expansion of the growth rate begins with $\gamma_1 \sim k_z$. Although the numerical calculations of this paper take us far beyond the linear regime, our result may give some indication of how Fig. 2 should look near the left-hand edge. We include the calculation here, as both references known to us give incorrect results [9,10]. (Reference [9] is at least essentially correct, but gets the coefficient wrong as well as being too brief to follow easily.)

We perturb Eq. (1) around a solution (2) for some ν. Thus

$$n = n_0(x - vt, y) + \delta n e^{(\gamma t + i k_z z)}, \quad v = 3\nu. \qquad \text{(A1)}$$

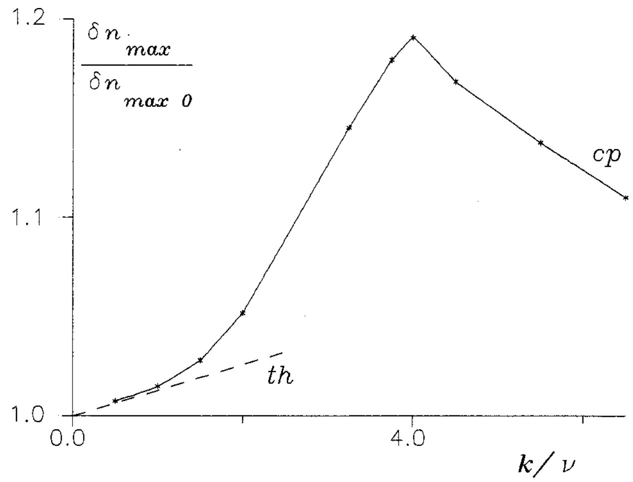

FIG. 2. The maximum value of $\delta n/\delta n_0$ of an unstable perturbation for fixed time of the run, $T=0.01$, as a function of $k_z\nu^{-1}$. We see that the most unstable normalized wave number $k_z\nu^{-1}$ is near 4. The value of $k_z\nu^{-1}$ used in our simulation was 0.5, as smaller length scales lead to numerical instabilities. We infer that the process described in the preceding figure is fastest when the length scale of the perturbation is eight times shorter than ours. The line on the left corresponds to the linear, small-wave-number limit given by Eq. (A8). This figure was feasible due to the extremely small value of T. For larger T and $k_z\nu^{-1}$ much larger than 0.5, numerical instabilities would set in. Variables are dimensionless. (Here th=theory and cp=computer generated.)

We obtain, after linearization,

$$L\,\delta n=-\gamma\delta n_x-3k_z^2\delta n, \tag{A2}$$

$$L=\partial_x^2[-v+6n_0+\partial_x^2]-3\partial_y^2. \tag{A3}$$

We now expand in k_z

$$\gamma=\gamma_1k_z+\gamma_2k_z^2+\cdots,\quad \delta n=\delta n_0+k_z\delta n_1+\cdots, \tag{A4}$$

and so

$$L\,\delta n_0=0. \tag{A5}$$

Usually expansions of this type lead to dispersion relations $\gamma=\gamma(k)$ at second order [5]. Equation (A5) is solved by

$$\delta n_0=\alpha n_x+\beta n_y+\rho(n_v-1/6).$$

The third function fails to vanish at infinity. There are two physical eigenmodes, n_x and n_y (we drop the zero subscript). This dualism often leads to one unstable and one stable mode, e.g., [22]. In Chap. 2 of [10], the authors claim to treat the stability of the n_y mode, and would have it be stable for our sign of the dispersion. However, their calculation is flawed, as the adjoint of L is miscalculated. In actual fact, only the n_x mode leads to a relation between γ and k_z in second order of the expansion. We now proceed to find this.

Take $\delta n_0=n_x$, leading to

$$\delta n_1=-\gamma_1(\sigma n_v+\beta),\quad \sigma+6\beta=1, \tag{A6}$$

in first order of (A2). Once again, the second component can be discarded as being unphysical. Thus $\beta=0$, $\sigma=1$. In second order, Eq. (A2) yields

$$L\,\delta n_2=\gamma_1^2n_{vx}-\gamma_2n_{xx}-3k_z^2n_x. \tag{A7}$$

We now need the eigenfunctions of the adjoint of the operator L. The only eigenfunction that vanishes at infinity is $\partial_x^{-1}n$. When we multiply Eq. (A7) by this function on the left and integrate twice by parts, the whole left-hand side vanishes. Using the form of n, Eq. (2), we obtain as a consistency condition on the right-hand side

$$\gamma_1=6\nu^{1/2}k_z, \tag{A8}$$

for the n_x mode, which in any case is the linear limit of our perturbation as used in Fig. 1. This augments the behavior found numerically, as there is a limit on the length of a perturbation in a simulation; see Fig. 2.

The linear growth rate might seem to be slightly too large. An explanation of how this can come about can be found in [22] (it concerns the periodic boundary conditions of a simulation). Nonlinear behavior takes over very soon [17].

Calculations such as the above can be rendered much more rigorous [23]. This will be the subject of a separate paper [24]. However, the treatment of [23] does validate values of γ_1 obtained by the above, less rigorous, calculation in a somewhat similar context in which the strongest secular terms only are removed. It transpires that a more rigorous treatment leaves γ_1 intact.

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